Stability of traveling waves in smectic-*C* liquid crystals

I. W. Stewart
Department of Mathematics, University of Strathclyde, Livingstone Tower, 26 Richmond Street, Glasgow G1 1XH, United Kingdom
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A stability analysis is carried out for a known exact traveling wave solution to a dynamic equation with sinusoidal nonlinearities, which arises in the study of smectic-*C* liquid crystals. The analysis is set in the usual context of traveling wave stability where suitably small perturbations may induce a phase shift to the original traveling wave. A control parameter β, which is related to the physical properties of the liquid crystal and the applied electric field, determines the possibility of a phase shift and characterizes the possible oscillatory or monotonic decay of the perturbations; these properties are derived from the spectrum of the perturbation equation. The consequences of these results are discussed for a smectic-*C* liquid crystal under the influence of an applied electric field tilted to the planes of the smectic layers and for a problem arising in chiral smectic-*C* liquid crystals. [S1063-651X(98)08505-5]

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I. INTRODUCTION

Traveling waves are known to arise in many applications of smectic liquid crystals. One of the most frequently occurring dynamic equations modeling such behavior is of the form

$$\phi_{zz}-\phi_t=\sin\phi+\beta \sin\phi \cos\phi\equiv F(\phi,\beta), \qquad (1.1)$$

where $\beta>0$ is a constant that depends on the physical parameters of the liquid-crystal problem being investigated. This equation has been derived and examined by many authors (for example, see Refs. [1–5]) who have exploited the well known exact traveling wave solution

$$\phi_0(z,t)=2\tan^{-1}\{\exp[(z-ct)\sqrt{\beta}]\}, \quad c=1/\sqrt{\beta}. \qquad (1.2)$$

The traveling wave (1.2) connects the stable state $\phi=0$ to the state $\phi=\pi$, which is metastable for $\beta>1$ but is generally unstable for $0<\beta<1$. In addition to the usual stability analysis in $L_2(\mathbb{R})$ for $\beta>1$, a restricted stability concept is introduced below (in Sec. IV) for $0<\beta<1$, which is common in the theory of traveling wave stability. This allows a more intricate discussion of the stability properties of Eq. (1.2) for $0<\beta<1$. The physical relevance of these results is briefly pointed out in Sec. V.

The solution (1.2) can be obtained by a phase plane analysis or, in a more general way, by using a nonlinear Painlevé analysis for partial differential equations. This latter method has recently been applied by the author [6] to produce static and complex solutions to Eq. (1.1) in addition to the solution (1.2). The aim of this article is to gain insight into the stability of Eq. (1.2) by examining the spectrum of a perturbation equation to Eq. (1.1). A similar approach to that employed below has been used in Refs. [7–9] for a version of Eq. (1.1) that models crossed field effects in nematic liquid crystals where the sinusoidal terms are effectively replaced by a cubic approximation in ϕ. For general details on traveling waves in nematic and smectic liquid crystals the reader should consult the reviews contained in Lam and Prost [10]. All of the above references relate to traveling waves for Eq. (1.1) in samples of liquid crystal that are infinite in the z direction; for a discussion on the effects of finite sample depth in chiral ferroelectric smectics see Maclennan, Clark, and Handschy [2]. The relationship to sample depth for a particularly special variant of Eq. (1.1) involving oscillating electric fields has been examined by Stewart, Carlsson, and Ardill [11].

To grasp some of the physics related to Eq. (1.1) we briefly mention the equations studied by Schiller, Pelzl, and Demus [4] and Cladis and van Saarloos [1] as typical examples: the other equations in the aforementioned references can be reformulated in a similar manner into the form of Eq. (1.1) by suitable rescalings. Liquid crystals are anisotropic fluids consisting of elongated molecules for which the long molecular axes locally adopt one common direction in space described by the unit vector **n**, called the director. Smectic-*C* liquid crystals are equidistant layered structures for which the director is tilted at a constant angle θ to the layer normal. The unit orthogonal projection **c** of **n** onto the smectic layers is commonly introduced. The orientation of **n** can easily be deduced from the orientation angle ϕ of **c**. For further details on the modeling of smectics the reader is referred to de Gennes and Prost [12] and Leslie, Stewart, and Nakagawa [13]. In simple planar layers of smectic *C* subjected to an electric field at a tilted angle α to the smectic layer normal the orientation angle ϕ satisfies the dynamic equation [4]

$$B\phi_{SS}-\lambda\phi_T=a \sin\phi+b \sin\phi \cos\phi, \qquad (1.3)$$

where T is time, S is a spatial variable, and the constants a and b are given by

$$a=\Delta\epsilon E^2\sin\theta \cos\theta \sin\alpha \cos\alpha, \qquad (1.4)$$

$$b=\Delta\epsilon E^2\sin^2\theta \sin^2\alpha. \qquad (1.5)$$

Here B is an elastic constant, λ is a viscosity coefficient, $\Delta\epsilon>0$ is the dielectric anisotropy of the liquid crystal, and E is the magnitude of the applied tilted electric field. Equation (1.3) can be rescaled via

$$t=T\,\frac{a}{\lambda},\quad z=S\sqrt{\frac{a}{B}},\tag{1.6}$$

which leads to Eq. (1.1) with

$$\beta=\frac{b}{a}=\tan\theta\,\tan\alpha.\tag{1.7}$$

The parameter β will control the possible types of stability behavior and the consequences of the results presented below will be related to Eq. (1.7) in Sec. V.

Cladis and van Saarloos [1] have discussed Eq. (1.1) in the context of the chiral smectic-C^* phase where $\Delta\epsilon<0$ and β ($=\Delta$ in [1], pp. 132–5) is replaced by

$$\beta=\frac{|\Delta\epsilon E|\theta}{4\pi P}.\tag{1.8}$$

Here P is the magnitude of the polarization vector **P**. Cladis and van Saarloos distinguished between three main behaviors for different values of β, namely, $0<\beta<1/2$, $1/2<\beta<1$, and $\beta>1$. It turns out that these are precisely the ranges of β that arise naturally in the work presented below. These results are also discussed further in Sec. V.

II. STABILITY

As mentioned in Sec. I, we discuss the stability of the exact traveling wave solution (1.2) to Eq. (1.1). We mainly consider the case for $\beta\geqslant1$ in Sec. III; the position for $0<\beta<1$ will be discussed in Sec. IV. As is common for such equations ([14], p. 158), and motivated by Eq. (1.2), Eq. (1.1) can be written in a moving coordinate frame by changing the variables to

$$t=t,\quad \tau=z-ct.\tag{2.1}$$

Equation (1.1) becomes

$$\phi_t=c\,\phi_\tau+\phi_{\tau\tau}-F(\phi,\beta).\tag{2.2}$$

We now consider solutions to Eq. (2.2) of the form

$$\phi(\tau,t)=\phi_0(\tau)+u(\tau,t),\tag{2.3}$$

where u is some small perturbation to the solution ϕ_0 in Eq. (1.2). By using the moving coordinate τ we can determine how the perturbation evolves in the moving reference frame associated with the traveling wave; if u decays, it will do so with respect to both the coordinates τ and t. This gives rise to the linearized perturbation equation for u,

$$u_t=u_{\tau\tau}+c\,u_\tau-\frac{\partial F}{\partial\phi_0}\,u,\tag{2.4}$$

where

$$\frac{\partial F}{\partial\phi_0}=\cos\phi_0+\beta\,\cos(2\phi_0).\tag{2.5}$$

Let A be the operator defined for $v\in L_2(\mathbb{R})$ by

$$-Av=v_{\tau\tau}+c\,v_\tau-\frac{\partial F}{\partial\phi_0}\,v.\tag{2.6}$$

Following the methods of Grindrod [15], the traveling wave ϕ_0 is said to be stable if the eigenvalues λ of A have $\mathrm{Re}(\lambda)\geqslant0$; the case when $\lambda=0$ (the "Goldstone mode") corresponds to those perturbations that result in a small phase shift in the moving coordinate frame of the traveling wave, that is, ϕ converges to $\phi_0(\tau+h)$ for some finite h as $t\to\infty$. When $\mathrm{Re}(\lambda)>0$ then all perturbations decay to zero: in fact, $\|v\|_{L^2}$ decays like $O(e^{-\delta t})$ whenever $\mathrm{Re}(\lambda)\geqslant\delta>0$ for any λ belonging to the spectrum ([15], p. 27). We therefore consider ϕ_0 to be stable whenever zero is an eigenvalue of A and the remainder of the spectrum of A lies in the complex half-space $\{\lambda:\mathrm{Re}(\lambda)\geqslant\delta\}$ for some $\delta>0$. Recall that the spectrum of A includes isolated eigenvalues of finite multiplicity together with the essential spectrum. For operators having nonconstant coefficients on the right hand side of Eq. (2.6) it is known that the essential spectrum of A can be located in the complex plane via standard results as follows (this is the one-dimensional version of the result contained in [15], p. 32): let B be the operator defined by

$$-Bv=Dv_{\tau\tau}-M(\tau)v_\tau-N(\tau)v,\tag{2.7}$$

where D is a positive constant, M and N are real bounded continuous functions and $M(\tau)\to M_\pm$, $N(\tau)\to N_\pm$ as $\tau\to\pm\infty$. Define

$$S_\pm=\{\lambda:Dk^2+ikM_\pm+N_\pm-\lambda=0;k\in\mathbb{R}\},\tag{2.8}$$

where $i^2=-1$. Then $S_\pm$ consists of two curves in the complex plane, each parametrized by k, which are symmetric about the real axis and are asymptotic to parabolas. The essential spectrum of B lies in the region between and including S_- and S_+ in the complex plane.

The central method mentioned by Grindrod for proving stability of ϕ_0 is to keep track of what is happening to the spectrum of A. The first step is to ensure that the essential spectrum of A lies to the right of the imaginary axis; if this is not the case, then, as shown in Sec. IV below, we may be able to "move" the essential spectrum to the right of the imaginary axis by restricting the possible class of allowable perturbations by only considering those perturbations that lie in a suitably weighted L_2 space. This leads to the second step, which is to show that the isolated eigenvalues of A are non-negative: this involves converting the equation to a self-adjoint form. These two steps provide sufficient information to conclude that each number in the spectrum of A has a non-negative real part and so when $\phi(\tau,0)-\phi_0(\tau)\in L_2(\mathbb{R})$ is sufficiently small [that is, the initial perturbation u in Eq. (2.3) is small at $t=0$],

$$\|\phi(\tau,t)-\phi_0(\tau+h)\|_{L_2}\to0\quad\text{as}\quad t\to\infty,\tag{2.9}$$

for some real constant h, that is, we have a form of L_2 stability. Here $\|\cdot\|_{L_2}$ denotes the usual L_2 norm (integrating with respect to τ). It turns out that we can always consider perturbations belonging to the space of $L_2(\mathbb{R})$ functions when $\beta\geqslant1$ while for $0<\beta<1$ we must consider more restricted perturbations lying in weighted L_2 spaces.

III. STABILITY FOR $\beta \geqslant 1$

A. Non-negativity of the eigenvalues

In this section we begin by showing that when $\beta \geqslant 1$ all the eigenvalues of A are real and non-negative by means of the techniques outlined in [15], pp. 118–121. Once it is known that $\lambda \geqslant 0$, the properties of the perturbation equation (2.6) can be discussed for both the discrete and continuous parts of the spectrum in order to determine the types of decay that small perturbations must obey for different values of λ. We recall that for any self-adjoint operator A acting in a Hilbert space H it is known that if λ belongs to the spectrum of A then either it is an eigenvalue or it may be regarded as an ''approximate'' eigenvalue since there is always a sequence $\{f_n\}$ (which does not necessarily converge to a function f) such that $\|f_n\|_H = 1$ and $\|(\lambda I - A) f_n\|_H \to 0$ ([16], p. 545). Thus it is important to examine the full spectrum to realize the behavior of the perturbations.

From the definition of F in Eq. (1.1) it follows that

$$\frac{\partial F}{\partial \phi_0} \to \beta \mp 1 \equiv F_{\pm} \quad \text{as} \quad \tau \to \pm\infty, \tag{3.1}$$

where $F_{\pm}$ represent the limits as $\tau \to \pm\infty$, respectively. Using Eqs. (2.6)–(2.8), the essential spectrum of A is contained between the curves:

$$S_{\pm} = \{\lambda : k^2 - ikc + F_{\pm} - \lambda = 0; k \in \mathbb{R}\}. \tag{3.2}$$

If we suppose that $\beta > 1$ then $F_{\pm} > 0$ and the essential spectrum lies completely in the right half of the complex plane, by Eq. (3.2). To prove that we have stability, that is, $\mathrm{Re}(\lambda) \geqslant 0$, it only remains to show that the isolated eigenvalues of the operator A have a non-negative real part. If such eigenvalues satisfy $\mathrm{Re}(\lambda) > 0$ then there is nothing to prove. Otherwise, suppose that $\mathrm{Re}(\lambda) \leqslant 0$ and that $u \in L_2$ satisfies the eigenvalue problem

$$0 = -Au + \lambda u = u_{\tau\tau} + c u_\tau + \left(\lambda - \frac{\partial F}{\partial \phi_0}\right) u. \tag{3.3}$$

From this linearized equation we can approximate $\partial F/\partial \phi_0$ by F_+ when $\tau \sim +\infty$ to show that u must decay exponentially to zero at least as $O(e^{-c\tau})$ since, by assumption, $\mathrm{Re}(\lambda) \leqslant 0$. Hence

$$y(\tau) = u(\tau) e^{(c/2)\tau} \tag{3.4}$$

will certainly decay exponentially to zero as $\tau \to \infty$; similarly, considering the linear equation (3.3) when $\tau \sim -\infty$, y will also tend to zero exponentially as $\tau \to -\infty$. A rigorous justification of these asymptotic properties follows from arguments similar to those detailed below in Sec. III B when the behavior related to the approximate eigenvalues for Eq. (3.3) is determined for $\tau \to \infty$ in Eq. (3.19). Hence $y \in L_2(\mathbb{R})$ and satisfies the self-adjoint problem (for $\tau \in \mathbb{R}$)

$$y_{\tau\tau} + \left(\lambda - \frac{\partial F}{\partial \phi_0} - \frac{c^2}{4}\right) y = 0 \tag{3.5}$$

with

$$y \to 0 \quad \text{as} \quad |\tau| \to \infty. \tag{3.6}$$

The above is a singular Sturm-Liouville boundary value problem for y and standard results show that all the eigenvalues λ for Eq. (3.5) are real, and hence any eigenvalue λ must be real for Eq. (3.3). [In L_2 the eigenfunctions corresponding to different eigenvalues in such singular problems are orthogonal in view of the boundary conditions (3.6) (see, for example, Birkhoff and Rota [17], p. 264) and this in turn shows that the eigenvalues must be real (this follows from the arguments used in Troutman [18], p. 272).] Multiplying Eq. (3.5) by y and integrating (by parts) over $\mathbb{R}$ gives

$$\lambda \int_{\mathbb{R}} y^2 d\tau = \int_{\mathbb{R}} \left\{ y_\tau^2 + \left(\frac{\partial F}{\partial \phi_0} + \frac{c^2}{4} \right) y^2 \right\} d\tau. \tag{3.7}$$

Differentiating Eq. (1.2) with respect to τ yields

$$\phi_{0\tau\tau\tau} + c\phi_{0\tau\tau} = \frac{\partial F}{\partial \phi_0} \phi_{0\tau}, \tag{3.8}$$

and therefore $\phi_{0\tau}$ is a solution to Eq. (3.3) when $\lambda = 0$. Hence $y = \psi(\tau) \equiv \phi_{0\tau}(\tau) e^{(c/2)\tau}$ satisfies Eq. (3.5) when $\lambda = 0$. Since ϕ_0 is never zero we know that $y = \psi(\tau)$ satisfies Eq. (3.5) with $\lambda = 0$ and so the expression $\partial F/\partial \phi_0 + c^2/4$ appearing in Eq. (3.7) can be replaced with $\psi_{\tau\tau}/\psi$. Equation (3.7) can then be rewritten as

$$\begin{aligned}
\lambda \int_{\mathbb{R}} y^2 d\tau &= \int_{\mathbb{R}} \left\{ y_\tau^2 + \frac{\psi_{\tau\tau}}{\psi} y^2 \right\} d\tau \\
&= \int_{\mathbb{R}} \left\{ y_\tau^2 - 2yy_\tau \frac{\psi_\tau}{\psi} + \frac{\psi_\tau^2}{\psi^2} y^2 \right\} d\tau \\
&= \int_{\mathbb{R}} \psi^2 \left[\frac{d}{d\tau} \left(\frac{y}{\psi} \right) \right]^2 d\tau.
\end{aligned} \tag{3.9}$$

It is now clear that λ must be non-negative and that $\lambda = 0$ only when y is a constant multiple of ψ, that is, when, by Eq. (3.4), u is a constant multiple of $\phi_{0\tau}$; when this is the case, such perturbations only result in a phase shift to the original wave ([15], p. 119), as in Eq. (2.9) for some $h \neq 0$. It now follows that if $u \in L_2(\mathbb{R})$ is sufficiently small then Eq. (2.9) is true and we can conclude that ϕ_0 is asymptotically stable. In the next section we examine the behavior of the eigenfunctions in order to determine the types of decay for the perturbations u.

B. Asymptotic behavior of perturbations

Returning to Eq. (3.3), we now know that $\mathrm{Re}(\lambda) \geqslant 0$ for Eq. (3.3) and therefore we can examine Eq. (3.5) for any λ in the spectrum and deduce the behavior of the corresponding $u(\tau)$ via the transformation (3.4). This approach has been adopted for equations similar to Eq. (1.1) containing a single sinusoidal term by Büttiker and Landauer [19] and Büttiker and Thomas [20]. Similar asymptotic methods for determining the continuous and discrete spectrum were used by Schlogl, Escher, and Berry [21] and here the transformations employed in [21] for an equation similar to Eq. (3.5) will be used.

Define

$$\xi=\tanh(\sqrt{\beta}\tau). \tag{3.10}$$

Then by Eq. (1.2)

$$\cos\phi_0=-\tanh(\sqrt{\beta}\tau), \tag{3.11}$$

$$\cos(2\phi_0)=2\tanh^2(\sqrt{\beta}\tau)-1, \tag{3.12}$$

and therefore, using Eq. (2.5), Eq. (3.5) can be written as, with $c=1/\sqrt{\beta}$,

$$y_{\tau\tau}+\left(\lambda-\frac{1}{4\beta}+\xi+\beta(1-2\xi^2)\right)y=0. \tag{3.13}$$

Since Eq. (3.13) is an unbounded linear operator in self-adjoint form its spectrum must be real (see Kreyszig [16]). As will be seen below, although λ is real the approximate eigensolutions to Eq. (3.3) may be complex valued functions. Define $q_{\pm}(\tau)$ and the (real) constants $b_{\pm}$ by

$$q_{\pm}(\tau)=\xi\mp 1+2\beta(1-\xi^2), \tag{3.14}$$

$$b_{\pm}=\lambda-\frac{1}{4\beta}-\beta\pm 1. \tag{3.15}$$

Then $q_{\pm}\to 0$ as $\tau\to\pm\infty$ respectively and Eq. (3.13) can be written as

$$y_{\tau\tau}+[b_{\pm}+q_{\pm}(\tau)]y=0. \tag{3.16}$$

The behavior of solutions for $\tau\to\infty$ will be investigated first, taking the plus signs in Eq. (3.16), which correspond to this behavior. Noting that $q_{+}(\tau)>0$ for $\beta\geqslant 1$ and $0\leqslant\tau<\infty$, direct integration reveals that

$$\int_0^\infty |q_{+}(\tau)|d\tau=\lim_{\tau\to\infty}\left[\frac{1}{\sqrt{\beta}}\ln[\cosh(\sqrt{\beta}\tau)]+2\sqrt{\beta}\tanh(\sqrt{\beta}\tau)-\tau\right]$$

$$=2\sqrt{\beta}-\frac{1}{\sqrt{\beta}}\ln(2)<\infty, \tag{3.17}$$

and hence the conditions that allow the application of standard asymptotic results for $\tau\to\infty$ are satisfied (see, for example, Hartman [22], p. 381 or de Brujin [23], p. 195). It can now be asserted that for $\tau\to\infty$ there are constants A_{+} and B_{+} such that

$$y(\tau)\sim A_{+}\exp(\sqrt{-b_{+}}\tau)+B_{+}\exp(-\sqrt{-b_{+}}\tau), \tag{3.18}$$

irrespective of the sign of b_{+}, and therefore by Eq. (3.4)

$$u(\tau)\sim A_{+}\exp\left(-\frac{1}{2\sqrt{\beta}}(1-\sqrt{-4\beta b_{+}})\tau\right)+B_{+}\exp\left(-\frac{1}{2\sqrt{\beta}}(1+\sqrt{-4\beta b_{+}})\tau\right). \tag{3.19}$$

[Notice that if λ had been negative then u would have to decay at least as $O(e^{-\tau/\sqrt{\beta}})$, justifying the statement after Eq. (3.3); A_{+} would have to be set as zero since $-4\beta b_{+}\geqslant 1$ for $\lambda<0$.] Similarly, $q_{-}(\tau)>0$ for $-\infty<\tau\leqslant 0$ and $\int_{-\infty}^{0}|q_{-}(\tau)|d\tau<\infty$ and therefore as $\tau\to-\infty$

$$y(\tau)\sim A_{-}\exp(\sqrt{-b_{-}}\tau)+B_{-}\exp(-\sqrt{-b_{-}}\tau) \tag{3.20}$$

and

$$u(\tau)\sim A_{-}\exp\left(-\frac{1}{2\sqrt{\beta}}(1-\sqrt{-4\beta b_{-}})\tau\right)+B_{-}\exp\left(-\frac{1}{2\sqrt{\beta}}(1+\sqrt{-4\beta b_{-}})\tau\right). \tag{3.21}$$

We require u to vanish for both $\tau\to\infty$ and $\tau\to-\infty$. Hence $B_{-}=0$ and one of the constants A_{+}, A_{-}, and B_{+} is a linear combination of the remaining two. For convenience we introduce the three constants

$$\lambda_1=\beta-1,\quad \lambda_2=\frac{1}{4\beta}+\beta-1,\quad \lambda_3=\beta+1. \tag{3.22}$$

Firstly, consider the case when $A_{+}\neq 0$. Then we require from Eq. (3.19) that $-4\beta b_{+}<1$, that is, $\lambda>\lambda_1$. Also, from Eq. (3.21), we need $-4\beta b_{-}>1$, that is, $\lambda<\lambda_3$. Hence

$$\lambda_1<\lambda<\lambda_3, \tag{3.23}$$

and these values of λ correspond to possible points in the continuous spectrum. It is easily seen from Eqs. (3.15) and (3.19) that when λ satisfies

$$\lambda_2<\lambda<\lambda_3 \tag{3.24}$$

then the approximate solutions $u(\tau)$ for this part of the spectrum consist of damped oscillations of period $\pi/\sqrt{\beta b_{+}}$ as $\tau\to\infty$ while for $\lambda_1<\lambda<\lambda_2$ they decay monotonically as $\tau\to\infty$. As $\tau\to-\infty$ there is always a monotonic decay for $\lambda<\lambda_3$.

The above results are true when the constant A_{+} is nonzero. The condition (3.23) is not the only possibility for bounded approximate solutions: if $A_{+}=0$ then, by Eq. (3.21), it is only known that $\lambda<\lambda_3$ and therefore it only remains to locate the isolated eigenvalues of finite multiplicity (which lie outside the essential spectrum) with their corresponding eigenfunctions, which we now discuss in the following using an argument similar to that used in [21]. By Eqs. (2.5), (3.11), and (3.12), Eq. (3.3) is

$$u_{\tau\tau}+cu_{\tau}+[\lambda+\xi+\beta(1-2\xi^2)]u=0. \tag{3.25}$$

With the transformation

$$\xi = \tanh(\sqrt{\beta}\tau) = 2x - 1 \tag{3.26}$$

and the value of c given in Eq. (1.2), Eq. (3.25) takes the form

$$x(1-x)\frac{d^2u}{dx^2} + \left(1 + \frac{1}{2\beta} - 2x\right)\frac{du}{dx} + \left(2 + \frac{\lambda - \beta - 1}{4\beta x} + \frac{\lambda - \beta + 1}{4\beta(1-x)}\right)u = 0. \tag{3.27}$$

Now insert the ansatz

$$u(x) = (1-x)^p x^q g(x), \tag{3.28}$$

where p and q are non-negative and seek solutions to Eq. (3.27) where $g(x)$ is a power series that is uniformly convergent at $x=0$ and $x=1$, that is, when $\tau \to \pm\infty$. The resulting equation for g is

$$x(1-x)\frac{d^2g}{dx^2} + \left(1 + \frac{1}{2\beta} + 2q - 2x(p+q+1)\right)\frac{dg}{dx} - \left(2pq - 2 + p(p+1) + q(q+1) - \frac{P(p,\lambda,\beta)}{1-x} - \frac{Q(q,\lambda,\beta)}{x}\right)g = 0 \tag{3.29}$$

with

$$P(p,\lambda,\beta) = p^2 - \frac{p}{2\beta} + \frac{1}{4\beta}(\lambda - \beta + 1), \tag{3.30}$$

$$Q(q,\lambda,\beta) = q^2 + \frac{q}{2\beta} + \frac{1}{4\beta}(\lambda - \beta - 1). \tag{3.31}$$

For g to be uniformly convergent at $x=0$ and $x=1$ it is evident from Eq. (3.29) that we must have

$$P = Q = 0. \tag{3.32}$$

The differential equation for g then becomes

$$x(1-x)\frac{d^2g}{dx^2} + [n - (l+m+1)x]\frac{dg}{dx} - lmg = 0, \tag{3.33}$$

where

$$l = p + q - 1, \tag{3.34}$$

$$m = p + q + 2, \tag{3.35}$$

$$n = 1 + \frac{1}{2\beta} + 2q. \tag{3.36}$$

The possible solutions to Eq. (3.33) that remain bounded as $x \to 1,0$ consist of the hypergeometric function ([24], 15.3.6, [25], 9.153),

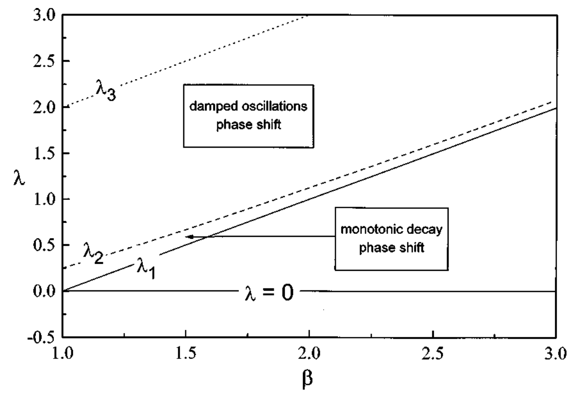

FIG. 1. Plots of λ_1, λ_2, and λ_3 given by Eq. (3.22) for $\beta \geqslant 1$. For points λ in the spectrum where $\lambda_1 < \lambda < \lambda_2$ there is the possibility of monotonic decay for perturbations in L_2 while for $\lambda_2 < \lambda < \lambda_3$ there may be damped oscillations of period $\pi/\sqrt{\beta b_+}$ as $\tau \to \infty$, these behaviors being superimposed upon the phase-shifted mode given by $\lambda = 0$.

$$g(x) = F(l,m,n;x) = \frac{\Gamma(n)\Gamma(n-l-m)}{\Gamma(n-l)\Gamma(n-m)} F(l,m,l+m-n+1;1-x) + (1-x)^{n-l-m}\frac{\Gamma(n)\Gamma(l+m-n)}{\Gamma(l)\Gamma(m)} \times F(n-l,n-m,n-l-m+1;1-x). \tag{3.37}$$

Therefore, by considering the behavior as $x \to 0$ or 1, in all relevant cases g will remain bounded whenever $\Gamma(l)$ [or $\Gamma(m)$] diverges, that is, whenever l is a nonpositive integer [24]. As in [21], when this is the case we obtain the eigenfunctions to Eq. (3.25) [via Eqs. (3.28) and (3.32)], which correspond to discrete eigenvalues λ. The first case to consider is $l = -1$, that is, $p + q = 0$. This implies that $p = q = 0$ (because of their assumed non-negativity in deriving Eqs. (3.29) to (3.33)] and then Eqs. (3.30) and (3.31) are inconsistent with the requirement (3.32). Hence $l = -1$ does not lead to a solution of Eq. (3.33). The only remaining possibility is $l = 0$, that is, $p + q = 1$. Here, subtracting Eq. (3.31) from (3.30) shows that $p = q$ since Eq. (3.32) must hold; thus $p = q = 1/2$. This forces Eqs. (3.30) and (3.31) to be consistent with Eq. (3.32) only if $\lambda = 0$. Since $l = 0$ forces $F \equiv 1$ it follows that generally $g = g_0 \equiv$ const and hence the required corresponding solution u_0 is given by

$$u_0(\tau) = g_0(1-x)^{1/2}x^{1/2} = \frac{g_0}{2}\,\mathrm{sech}(\sqrt{\beta}\tau). \tag{3.38}$$

This solution for the isolated eigenvalue $\lambda = 0$ is simply a multiple of $\phi_{0\tau}$, the derivative of the traveling wave solution in Eq. (1.2), as can be checked directly. Thus the only discrete mode is for $\lambda = 0$ whose eigenfunction represents a phase shift to the original traveling wave, as mentioned in Sec. III A. Notice that $\lambda = 0$ is always outside the continuous spectrum for $\beta > 1$. These results are summarized in Fig. 1. Further use of Eq. (3.38) will be made in the next section.

IV. STABILITY FOR $0<\beta<1$

A. Non-negativity of the eigenvalues

For $0<\beta<1$ Eq. (3.1) is still valid, the only difference being that $F_->0$ while $F_+<0$. Equation (3.2) then shows that the essential spectrum can lie partly in the left half of the complex plane and therefore some part of the spectrum for problem (3.3) may have a negative real part, indicating possible instability. Nevertheless, if the perturbations are more restricted and belong to a weighted L_2 space, as indicated below, then the essential spectrum arising from this restriction can lie in the right half of the complex plane. We show that it is possible to have stability for $0<\beta<1$ when suitably restricted perturbations are considered.

Following the methods outlined in [15], pp. 31–34, we require perturbations $u(\tau)$ to satisfy

$$\|u\|_{L_2^\gamma}=\left\{\int_{\mathbb{R}} e^{2\gamma\tau}|u(\tau)|^2 d\tau\right\}^{1/2}<\infty, \tag{4.1}$$

where $\gamma>0$ is to be determined. Here the weighting function is the exponential term in Eq. (4.1); when $\gamma=0$ we recover the usual space of L_2 functions. As in [15], define the operator A^* to be the restriction of A to the space L_2^γ where γ can be sought such that the essential spectrum of A^* lies to the right of the imaginary axis, that is, we restrict the perturbations to lie in a suitable space L_2^γ. We first notice that A^* acting in the space L_2^γ is equivalent to $e^{\gamma\tau}A^*(ve^{-\gamma\tau})$ acting on functions $v\in L_2$. This leads us to define the operator W by

$$W=e^{\gamma\tau}A^*\left(\frac{\cdot}{e^{\gamma\tau}}\right), \tag{4.2}$$

which is the operator on L_2 induced by A^*. The spectrum of W coincides with that of A^* and, further,

$$-Wv=v_{\tau\tau}+(c-2\gamma)v_\tau+\left(\gamma^2-c\gamma-\frac{\partial F}{\partial\phi_0}\right)v. \tag{4.3}$$

Now we can use the same argument for obtaining the modified version of Eq. (3.2) using Eqs. (2.7) and (2.8) to find that the essential spectrum of W, and hence that of A^*, lies between the curves

$$S_\pm=\{\lambda:k^2+(2\gamma-c)ki+(F_\pm-\gamma^2+c\gamma)-\lambda=0;k\in\mathbb{R}\}. \tag{4.4}$$

Thus the essential spectrum of A^* is contained in the right half plane

$$\mathrm{Re}(\lambda)\geq F_+-\gamma^2+c\gamma, \tag{4.5}$$

with F_+ given by Eq. (1.2). The problem is to now find values of γ such that $\mathrm{Re}(\lambda)\geq 0$, which will allow us to transform the problem to self-adjoint form. It will be sufficient for our purposes to notice that, for any $0<\beta<1$, we have

$$F_+-\gamma^2+c\gamma=\frac{1}{\beta}\left(\beta-\frac{1}{2}\right)^2=\lambda_2\geq 0$$

$$\text{whenever}\quad \gamma=\frac{c}{2}=\frac{1}{2\sqrt{\beta}}. \tag{4.6}$$

This choice of γ ensures that $\mathrm{Re}(\lambda)\geq 0$ for λ in the essential spectrum and allows us to use the transformation (3.4) to convert the problem to self-adjoint form. An identical argument to that given in Sec. III A then shows that the spectrum is real and that $\lambda\geq 0$ for all λ belonging to the spectrum. From Eq. (4.5) it is certainly true that the essential spectrum is contained in the interval $[\lambda_2,\infty)$. It follows that for $0<\beta<1$ the traveling wave ϕ_0 is stable to perturbations in L_2^γ with γ given by Eq. (4.6).

B. Asymptotic behavior of perturbations

Throughout this section we assume γ is given by Eq. (4.6). Using the results in Sec. III B, we know that $u\in L_2^\gamma$ leads to considering the behavior of $y=u(\tau)e^{\gamma\tau}$ as $\tau\to\pm\infty$. For any λ belonging to the essential spectrum (which is contained in $[\lambda_2,\infty)$) the ''approximate'' solutions have monotonic decay. For (discrete) eigenvalues of finite multiplicity the required results are given in Eqs. (3.18) and (3.20). For u to be contained in L_2^γ for such isolated eigenvalues it is readily seen from Eq. (3.18) that we need $b_+<0$ and $A_+=0$, which means that

$$\lambda<\lambda_2. \tag{4.7}$$

Similarly, from Eq. (3.20) we require $B_-=0$ and $b_-<0$, leading to the restriction that $\lambda<1/4\beta+\beta+1$. Since both equations (3.18) and (3.20) must be integrable we need condition (4.7) to hold in both cases: this provides an upper bound for any considered possible isolated eigenvalues. Notice that $\lambda_2=0$ at $\beta=1/2$ and $\lambda_2>0$ for $\beta\neq 1/2$.

The argument in the latter part of Sec. III B, from Eq. (3.25) onwards, can be employed here to show that the only possible eigenfunction is that given by $u_0(\tau)$ in Eq. (3.38) when $\lambda=0$. To check that such a perturbation is in L_2^γ it is necessary to look at the behavior as $\tau\to\pm\infty$; clearly, $u_0(\tau)e^{\gamma\tau}\to 0$ as $\tau\to-\infty$ while

$$u_0(\tau)e^{\gamma\tau}\sim e^{-(\tau/2\sqrt{\beta})(2\beta-1)} \tag{4.8}$$

as $\tau\to\infty$. Therefore $u_0\in L_2^\gamma$ provided $\beta>1/2$ in which case u_0 induces, as before, a phase shift to the original solution. For $0<\beta\leq 1/2$ only the essential spectrum is available; this indicates that $\lambda=0$ is no longer an isolated eigenvalue and therefore perturbations must decay monotonically to zero in L_2^γ (isolated eigenvalues often vanish when working in weighted spaces [15]). This also means, in contrast to the previous cases, that there is no phase shift imposed upon the original solution for $0<\beta\leq 1/2$ when perturbations belong to L_2^γ. These results are summarized in Fig. 2.

V. DISCUSSION

It has been shown that the traveling wave solution (1.2) is linearly stable for $\beta\geq 1$ and that there is always a small

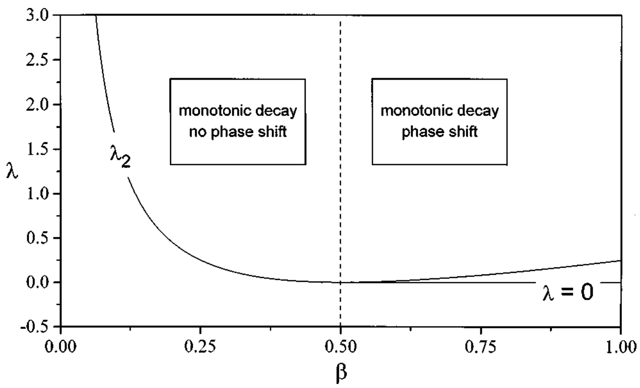

FIG. 2. For $1/2<\beta<1$ there is the possibility that perturbations belonging to L_2^γ consist of monotonic decay superimposed upon the phase-shifted solution at $\lambda=0$ when $\lambda>\lambda_2$. For $0<\beta\leqslant 1/2$ there cannot be a phase-shifted solution and therefore all perturbations in L_2^γ decay monotonically to zero.

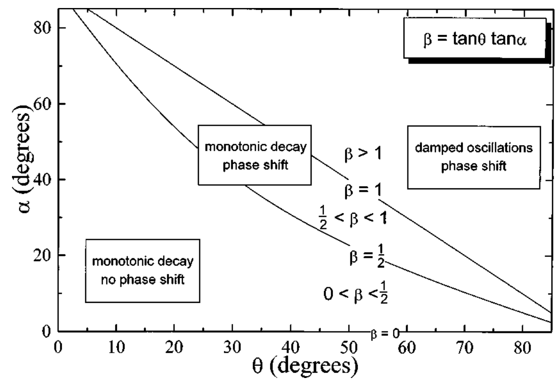

FIG. 3. Contours as indicated of the parameter β given by Eq. (1.7) for $0\leqslant\alpha\leqslant 85°$ and $0\leqslant\theta\leqslant 85°$. The possible effects of suitably small perturbations are schematically summarized (see Secs. III and IV in the text). The possibility of an induced phase shift to the original traveling wave (1.2) and the decay properties of the perturbations are as shown. Perturbations u belong to L_2 for $\beta\geqslant 1$ and to L_2^γ for $\beta<1$.

phase shift to this original solution when it is subjected to small perturbations belonging to $L_2(\mathbb{R})$. The possible decay properties of such perturbations have been characterized in Fig. 1 via an investigation of the possible spectrum arising from the perturbation equation. In addition to the phase shift a superimposed monotonic and/or oscillatory decay is possible with these perturbations. For $0<\beta<1$ the solution (1.2) is stable when the small perturbations belong to L_2^γ [defined in Eq. (4.1)] with $\gamma=1/(2\sqrt{\beta})$. When $1/2<\beta<1$ such perturbations induce a small phase shift while for $0<\beta\leqslant 1/2$ all perturbations decay to zero with, in contrast to the other cases, no phase shift imposed upon the original traveling wave. These decay properties are characterized in Fig. 2.

These results can be interpreted from the physical point of view in relation to the work by Schiller *et al.* [4] via equations (1.3)–(1.7). Figure 3 shows contour plots for Eq. (1.7) at $\beta=0$, 1/2 and 1 for $0\leqslant\alpha\leqslant 85°$ and $0\leqslant\theta\leqslant 85°$. The three ranges of β considered in Secs. III and IV lead to three regions in the $\theta\alpha$ parameter space as indicated in the figure. For each of these regions the possible effects of suitably small perturbations are summarized: the original traveling wave may or may not have an induced phase shift, as indicated schematically in the figure; the possibilities for monotonic and/or oscillatory decay of the perturbations are also shown.

Cladis and van Saarloos [1] have discussed marginal stability for β given by Eq. (1.8) for the same regimes of positive values as discussed above. In summary, their results indicate that for $0<\beta<1/2$, the more relevant solution for the problem they investigate is not the exact solution (1.2) but rather a marginal stability solution with a wave speed of $2(1-\beta)^{1/2}$, relevant for small fields. Clearly, when $\beta=1/2$ this wave speed coincides with that given by Eq. (1.2). The results presented in this present article for $0<\beta<1/2$ indicate that the perturbation decays monotonically, that is, Eq. (1.2) is stable in $L_2^\gamma(\mathbb{R})$. This apparent inconsistency with the work in [1] is resolved by other physical and mathematical arguments [26] where the value of the velocity is ''selected'' by some type of dynamical mechanism when $\beta<1$, as mentioned in [1]. Restricting the choice of perturbations so that they belong to $L_2^\gamma(\mathbb{R})$ limits the allowed physical forms of perturbation: the large τ behavior of the original perturbation must be better controlled so that it is in $L_2^\gamma(\mathbb{R})$. This means that Eq. (1.2), although mathematically a solution, is not in general the physically relevant solution for $\beta<1/2$ for the problem in [1] unless the original perturbations to Eq. (1.2) always lie in $L_2^\gamma(\mathbb{R})$. A reported preliminary analysis by Cladis and van Saarloss (of [40] of Ref. [1]) shows that $1/2<\beta<1$ corresponds to a ''case II marginal stability,'' as discussed by Ben-Jacob *et al.* [27]. In this regime the exact solution (1.2) is the physically relevant one. For $\beta>1$ the solution (1.2) describes a front propagating from a metastable state into a stable state and this exact solution is again the physically meaningful one. For $\beta\geqslant 1/2$ the stability results reported here are therefore consistent with those in [1] and there will always be a phase shift to Eq. (1.2) upon perturbation for $\beta>1/2$ with, by the comments after Eq. (4.8), no phase shift for $\beta=1/2$. The decay properties of the perturbations remain as discussed in Secs. III and IV, namely, monotonic decay for $1/2<\beta<1$ and monotonic and/or oscillatory decay for $\beta>1$. Front propagation and marginal stability has been discussed further by van Saarloos, Hecke, and Holyst [3].

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